

An improved Kalai-Kleitman bound for the diameter of a polyhedron

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Abstract

Kalai and Kleitman [6] established the bound $n^{\log(d)+2}$ for the diameter of a d -dimensional polyhedron with n facets. Here we improve the bound slightly to $(n-d)^{\log(d)}$.

1 Introduction

A d -polyhedron P is a d -dimensional set in \mathbb{R}^d that is the intersection of a finite number of half-spaces of the form $H := \{x \in \mathbb{R}^d : a^T x \leq \beta\}$. If P can be written as the intersection of n half-spaces H_i , $i = 1, \dots, n$, but not fewer, we say it has n facets and these facets are the faces $F_i = P \cap H_i$, $i = 1, \dots, n$, each affinely isomorphic to a $(d-1)$ -polyhedron with at most $n-1$ facets. We then call P a (d, n) -polyhedron.

We say $v \in P$ is a *vertex* of P if there is a half-space H with $P \cap H = \{v\}$. (A polyhedron is *pointed* if it has a vertex, or equivalently, if it contains no line.) Two vertices v and w of P are *adjacent* (and the set $[v, w] := \{(1-\lambda)v + \lambda w : 0 \leq \lambda \leq 1\}$ an *edge* of P) if there is a half-space H with $P \cap H = [v, w]$. A *path of length k* from vertex v to vertex w in P is a sequence $v = v_0, v_1, \dots, v_k = w$ of vertices with v_{i-1} and v_i adjacent for $i = 1, \dots, k$. The *distance* from v to w is the length of the shortest such path and is denoted $\rho_P(v, w)$, and the *diameter* of P is the largest such distance,

$$\delta(P) := \max\{\rho_P(v, w) : v \text{ and } w \text{ vertices of } P\}.$$

We define

$$\Delta(d, n) := \max\{\delta(P) : P \text{ a } (d, n)\text{-polyhedron}\}$$

and seek an upper bound on $\Delta(d, n)$. It is not hard to see that $\Delta(d, \cdot)$ is monotonically non-decreasing. Also, the maximum above can be attained by a *simple* polyhedron, one where each vertex lies in exactly d facets. See, e.g., Klee and Kleinschmidt [12] or Ziegler [18]. A related paper, Ziegler [19], gives the history of the Hirsch conjecture that $\Delta_b(d, n) \leq n-d$, where $\Delta_b(d, n)$ is defined as above but for bounded polyhedra.

In the last few years, there has been an explosion of papers related to the diameters of polyhedra and related set systems. Santos [14] found a counterexample to the Hirsch

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conjecture, later refined by Matschke, Santos, and Weibel [13]. Eisenbrand, Hähnle, Razborov, and Rothkoss [3] showed that a slightly improved Kalai-Kleitman bound, $n^{\log(d)+1}$, held for a very general class of set families abstracting properties of the vertices of d -polyhedra with n facets, which included the ultraconnected set families considered earlier by Kalai [4]. (This improved bound, for polyhedra, was presented first in Kalai [5].) Another class of set families was introduced by Kim [7]; adding various properties gave families for which this bound held, or other families where the maximum diameter grew exponentially. The latter result is due to Santos [15]. Earlier combinatorial abstractions of polytopes include the abstract polytopes of Adler and Dantzig [1] (these satisfy the Hirsch conjecture for $n - d \leq 5$) and the duoids of [16, 17] (these have a lower bound on their diameter growing quadratically with $n - d$). We also mention the nice overview articles of Kim and Santos [8] (pre-counterexample) and De Loera [2] (post-counterexample).

Our bound $(n-d)^{\log(d)}$ fits better with the Hirsch conjecture and is tight for dimensions 1 and 2. Also, more importantly, it is invariant under linear programming duality. A pointed d -polyhedron with n facets can be written as $\{x \in \mathbb{R}^d : Ax \leq b\}$ for some $n \times d$ matrix A of full rank and some n -vector b . Choosing an objective function $c^T x$ for $c \in \mathbb{R}^d$ gives the linear programming problem $\max\{c^T x : Ax \leq b\}$, whose dual is $\min\{b^T y : A^T y = c, y \geq 0\}$. The feasible region for the latter is affinely isomorphic to a pointed polyhedron of dimension at most $n - d$ with at most n facets, and equality is possible. Hence duality switches the dimensions d and $n - d$.

2 Result

We prove

Theorem 1 *For $1 \leq d \leq n$, $\Delta(d, n) \leq (n - d)^{\log(d)}$, with $\Delta(1, 1) = 0$.*

(All logarithms are to base 2; note that $(n - d)^{\log(d)} = d^{\log(n-d)}$ as both have logarithm $\log(d) \cdot \log(n - d)$. We use this in the proof below.)

The key lemma is due to Kalai and Kleitman [6], and was used by them to prove the bound $n^{\log(d)+2}$. We give the proof for completeness.

Lemma 1 *For $2 \leq d \leq \lfloor n/2 \rfloor$, where $\lfloor n/2 \rfloor$ is the largest integer at most $n/2$,*

$$\Delta(d, n) \leq \Delta(d - 1, n - 1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2.$$

Proof: Let P be a simple (d, n) -polyhedron and v and w two vertices of P with $\delta_P(v, w) = \Delta(d, n)$. We show there is a path in P from v to w of length at most the right-hand side above. If v and w both lie on the same facet, say F , of P , then since F is affinely isomorphic to a $(d - 1, m)$ -polyhedron with $m \leq n - 1$, we have $\rho_P(v, w) \leq \rho_F(v, w) \leq \Delta(d - 1, m) \leq \Delta(d - 1, n - 1)$ and we are done.

Otherwise, let k_v be the largest k so that there is a set \mathcal{F}_v of at most $\lfloor n/2 \rfloor$ facets with all paths of length k from v meeting only facets in \mathcal{F}_v . This exists since all paths of length 0 meet only d facets (those containing v), whereas paths of length $\delta(P)$ can meet all n facets of P . Define k_w and \mathcal{F}_w similarly. We claim that $k_v \leq \Delta(d, \lfloor n/2 \rfloor)$ and similarly for k_w . Indeed, let $P_v \supseteq P$ be the (d, m_v) -polyhedron ($m_v = |\mathcal{F}_v| \leq \lfloor n/2 \rfloor$) defined by just those linear inequalities corresponding to the facets in \mathcal{F}_v . Consider any vertex t of P a distance k_v from v , so there is a shortest path from v to t of length k_v meeting only

facets in \mathcal{F}_v . But this is also a shortest path in P_v , since if there were a shorter path, it could not be a path in P , and thus must meet a facet not in \mathcal{F}_v , a contradiction. So

$$k_v = \delta_{P_v}(v, t) \leq \Delta(d, m_v) \leq \Delta(d, \lfloor n/2 \rfloor).$$

Now consider the set \mathcal{G}_v of facets that can be reached in at most $k_v + 1$ steps from v , and similarly \mathcal{G}_w . Since both these sets contain more than $\lfloor n/2 \rfloor$ facets, there must be a facet, say G , in both of them. Thus there are vertices t and u in G and paths of length at most $k_v + 1$ from v to t and of length at most $k_w + 1$ from w to u . Then

$$\begin{aligned} \Delta(d, n) &= \rho_P(v, w) \\ &\leq \rho_P(v, t) + \rho_G(t, u) + \rho_P(w, u) \\ &\leq k_v + 1 + \Delta(d - 1, n - 1) + k_w + 1 \\ &\leq \Delta(d - 1, n - 1) + 2\Delta(d, \lfloor n/2 \rfloor) + 2, \end{aligned}$$

since, as above, G is affinely isomorphic to a $(d - 1, m)$ -polyhedron with $m \leq n - 1$. \square

Proof of the theorem: This is by induction on $d + n$. The result is trivial for $n = d$, since there can be only one vertex. Next, the right-hand side gives 1 for $d = 1$ ($n = 2$) and $n - 2$ for $d = 2$, which are the correct values. For $d = 3$, it gives $(n - 3)^{\log(3)}$, which is greater than the correct value $n - 3$ established by Klee [9, 10, 11]. (We could make the proof more self-contained by establishing the $d = 3$ case from the lemma: a general argument deals with $n \geq 13$, but then there are seven more special cases to check.) Below we will give a general inductive step for the case $d \geq 4$, $n - d \geq 8$. Also, the result clearly holds by induction if $n < 2d$, since then any two vertices lie on a common facet, so their distance is at most $\Delta(d - 1, n - 1)$. The remaining cases are $d = 4$, $8 \leq n \leq 11$; $d = 5$, $10 \leq n \leq 12$; $d = 6$, $12 \leq n \leq 13$; and $d = 7$, $n = 14$. All these cases can be checked easily using the lemma, the equation $\Delta(d, d) = 0$, and the equations $\Delta(5, 6) = \Delta(4, 5) = \Delta(3, 4) = \Delta(2, 3) = 1$.

Now we deal with the case $d \geq 4$, $n - d \geq 8$. For this, $\log(n - d) \geq 3$, so we have

$$\begin{aligned} \Delta(d, n) &\leq \Delta(d - 1, n - 1) + 2 \cdot \Delta(d, \lfloor n/2 \rfloor) + 2 \\ &\leq (d - 1)^{\log(n - d)} + 2 \cdot d^{\log(n/2 - d)} + 2 \\ &\leq \left(\frac{d - 1}{d} \right)^{\log(n - d)} d^{\log(n - d)} + 2 \cdot d^{\log((n - d)/2)} + 2 \\ &\leq \left(\frac{d - 1}{d} \right)^3 d^{\log(n - d)} + \frac{2}{d} \cdot d^{\log(n - d)} + 2 \\ &= \left(1 - \frac{3}{d} + \frac{3}{d^2} - \frac{1}{d^3} + \frac{2}{d} \right) d^{\log(n - d)} + 2 \\ &\leq \left(1 - \frac{1}{d} + \frac{3}{4d} - \frac{1}{d^3} \right) d^{\log(n - d)} + 2 \\ &\leq d^{\log(n - d)} - \frac{1}{4d} \cdot d^{\log(n - d)} - \frac{1}{d^3} \cdot d^{\log(n - d)} + 2 \\ &\leq d^{\log(n - d)}, \end{aligned}$$

since each of the subtracted terms is at least one. This completes the proof. \square

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